BIFURCATION LOCI OF EXPONENTIAL MAPS AND QUADRATIC POLYNOMIALS: LOCAL CONNECTIVITY, TRIVIALITY OF FIBERS, AND DENSITY OF HYPERBOLICITY

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ABSTRACT. We study the bifurcation loci of quadratic (and unicritical) polynomials and exponential maps. We outline a proof that the exponential bifurcation locus is connected; this is an analog to Douady and Hubbard's celebrated theorem that (the boundary of) the Mandelbrot set is connected.

For these parameter spaces, a fundamental conjecture is that hyperbolic dynamics is dense. For quadratic polynomials, this would follow from the famous stronger conjecture that the bifurcation locus (or equivalently the Mandelbrot set) is locally connected. It turns out that a formally slightly weaker statement is sufficient, namely that every point in the bifurcation locus is the landing point of a parameter ray.

For exponential maps, the bifurcation locus is not locally connected. We describe a different conjecture (triviality of fibers) which naturally generalizes the role that local connectivity has for quadratic or unicritical polynomials.

1. BIFURCATION LOCI AND STABLE COMPONENTS

The family of *quadratic polynomials* $p_c \colon z \mapsto z^2 + c$, parametrized by $c \in \mathbb{C}$, contains, up to conformal conjugacy, exactly those polynomials with only a single, simple, critical value (at c). Hence this family is the simplest parameter space in the dynamical study of polynomials, and has correspondingly received much attention during the last two decades. Similarly, *exponential maps* $E_c \colon z \mapsto e^z + c$ are, up to conformal conjugacy, the only transcendental entire functions with only one singular value (the asymptotic value at c).[1] This simplest transcendental parameter space has likewise been studied intensively since the 1980s.

In the following, we will treat these parameter spaces in parallel, unless explicitly stated otherwise; we will write f_c for p_c or E_c. Following Milnor, we write $f_c^{\circ n}$ for the

[1] Often the parametrization $z \mapsto \lambda e^z$ with $\lambda \in \mathbb{C}^*$ is used instead. This has the asymptotic value at 0 and is conformally equivalent to E_c iff $\lambda = e^c$. This has the advantage that two maps $z \mapsto \lambda e^z$ and $z \mapsto \lambda' e^z$ are conformally conjugate iff $\lambda' = \lambda$. We prefer the parametrization as E_c not only for the analogy to the quadratic family, but also because all maps E_c have the same asymptotics near infinity, and because parameter space is simply connected, which leads to a more natural combinatorial description.
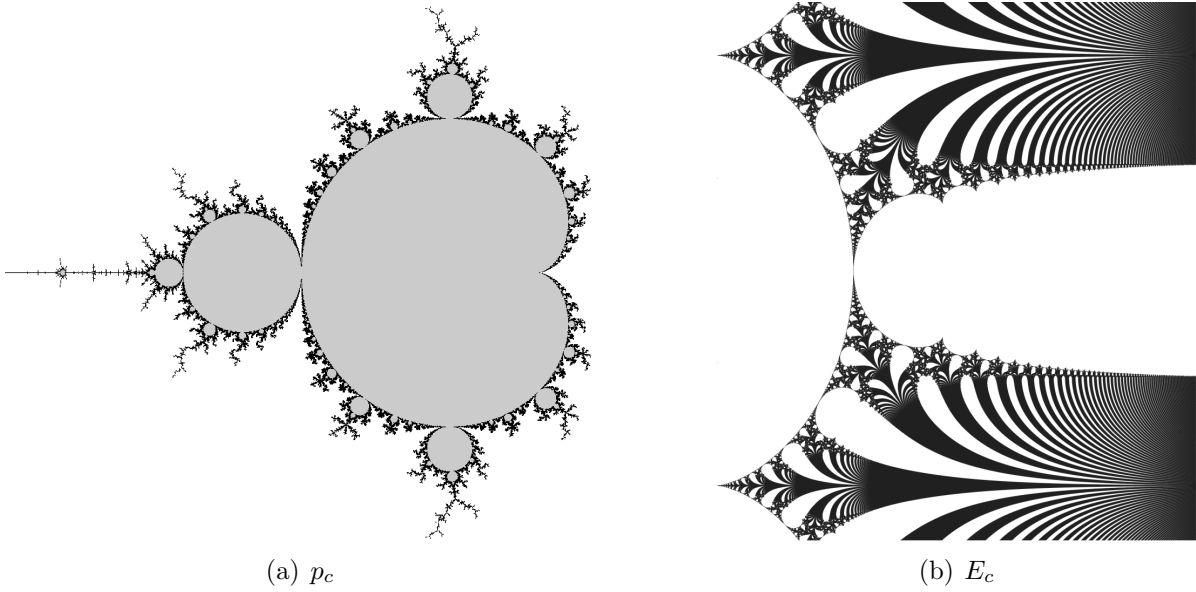
(a) p_c (b) E_c

FIGURE 1. The parameter spaces of quadratic polynomials p_c (left) and exponential maps E_c (right); the bifurcation loci are drawn in black. Unbounded hyperbolic components are white, bounded hyperbolic components are gray (these do not exist for exponentials).

n-th iterate of f_c. The map f_c is called *stable* if, for c' sufficiently close to c, the maps f_c and $f_{c'}$ are topologically conjugate on their Julia sets, and the conjugacy depends continuously on the parameter c' (the former condition implies the latter in our setting). We denote by $\mathcal{R}$ the *locus of stability*; that is, the (open) set of all $c \in \mathbb{C}$ so that f_c is stable. The set $\mathcal{R}$ is open and dense in $\mathbb{C}$ [MSS, EL].

A *hyperbolic component* is a connected component of $\mathcal{R}$ in which every map f_c has an attracting periodic orbit of constant period. Within any non-hyperbolic component of $\mathcal{R}$, all cycles of f_c would have to be repelling. One of the fundamental conjectures of one-dimensional holomorphic dynamics is the following.

Conjecture 1 (Hyperbolicity is Dense). *Every component of $\mathcal{R}$ is hyperbolic. (Equivalently, hyperbolic dynamics is open and dense in parameter space.)*

Hyperbolic components — both in the quadratic and in the exponential family — are completely understood in terms of their combinatorics [DH, S2, RS1]. The complement $\mathcal{B} := \mathbb{C} \setminus \mathcal{R}$ is called the *bifurcation locus*. Since $\mathcal{R}$ is open and dense in $\mathbb{C}$, $\mathcal{B}$ is closed and has no interior points. The bifurcation locus is extremely complicated (see Figure 1).

Theorem 2 (Bifurcation Loci Connected). *$\mathcal{B}$ is a connected subset of $\mathbb{C}$.*

For quadratic polynomials, $\mathcal{B}$ is the boundary of the famous Mandelbrot set $\mathcal{M}$, and Theorem 2 is equivalent to the fundamental theorem of Douady and Hubbard that $\mathcal{M}$ or equivalently $\partial\mathcal{M}$ is connected [DH, Exposé VIII.I]. For exponential maps, connectivity of the bifurcation locus is new [RS1, Theorem 1.1]; we outline a proof below.

To study the bifurcation locus, it is useful to consider the *escape locus*

$$\mathcal{I} := \{c \in \mathbb{C} \colon f_c^{\circ n}(c) \to \infty \text{ as } n \to \infty\} \ .$$

The set $\mathcal{I}$ decomposes naturally into a disjoint union of *parameter rays* and their endpoints (see below), and the ultimate goal is to describe $\mathcal{B}$ in terms of these rays. The structure of $\mathcal{I}$ and the parameter rays is well-understood, and conjecturally, every point in $\mathcal{B}$ is the landing point of a parameter ray or, in the case of exponential maps, on a parameter ray itself. This is in analogy to the dynamical planes of f_c, where the Julia sets are often studied using the simpler structure of the Fatou set or of the set of points that converge to ∞ under iteration. We show below (Theorem 11) that, for the quadratic family, the conjecture that every point in $\mathcal{B}$ is the landing point of a ray is equivalent to the famous open question of local connectivity of the Mandelbrot set; we then reformulate this conjecture in a uniform way for quadratic polynomials and exponentials.

A fundamental difference between the polynomials p_c and the exponential maps E_c is their behavior near ∞: every p_c has a superattracting fixed point at ∞ which attracts a neighborhood of ∞ in the Riemann sphere, while every E_c has an essential singularity at ∞ and the set of points converging to ∞ is extremely complicated [SZ]. This implies that in the parameter space of quadratic polynomials we have $\mathcal{I} = \mathbb{C} \setminus \mathcal{M}$ and there is a unique conformal isomorphism $\Phi \colon \mathcal{I} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ with $\Phi(c)/c \to 1$ as $c \to \infty$ (where $\mathbb{D}$ is the complex unit disk); the map Φ was constructed by Douady and Hubbard [DH, Exposé VIII.I] in their proof of connectivity of the Mandelbrot set. A parameter ray is defined as the preimage of a radial line in $\mathbb{C} \setminus \overline{\mathbb{D}}$ under Φ. On the other hand, the set $\mathcal{I}$ for exponential maps has a much richer topological structure [FRS]: it is the disjoint union of uncountably many curves (*parameter rays*) with or without endpoints; each parameter ray (possibly with its endpoint) is a path component of $\mathcal{I}$. More precisely, every path component of $\mathcal{I}$ is a curve $G_{\underline{s}} \colon (0, \infty) \to \mathcal{I}$ or $G_{\underline{s}} \colon [0, \infty) \to \mathcal{I}$, both times with $G_{\underline{s}}(t) \to \infty$ as $t \to \infty$.[2] The index $\underline{s}$ distinguishes different parameter rays: it is a sequence of integers and is called the *external address* of $G_{\underline{s}}$. Different rays have different external addresses, and the set of allowed external addresses can be described explicitly [FS, FRS]. We call the image $G_{\underline{s}}(0, \infty)$ the *parameter ray* at external address $\underline{s}$, and $G_{\underline{s}}(0)$ its *endpoint* (if it exists). Let $\mathcal{I}_R$ be the union of all parameter rays and $\mathcal{I}_E$ be the set of all endpoints in $\mathcal{I}$. We say that a parameter ray *lands* if $\lim_{t \searrow 0} G_{\underline{s}}(t)$ exists (note that many parameter rays $G_{\underline{s}}$ land at non-escaping parameters, and others might not land at all, but $\mathcal{I}_E$ consists only of those landing points that are in $\mathcal{I}$).

It is a consequence of the "λ-lemma" from [MSS] that $\mathcal{B} = \partial \mathcal{I}$ both for quadratic polynomials and for exponential maps; see e.g. [R1, Lemma 5.1.5]. In particular, the bifurcation locus of quadratic polynomials is a compact subset of $\mathbb{C}$, while for exponential maps it is unbounded.

We will also use the *reduced bifurcation locus*

$$\mathcal{B}^* := \mathcal{B} \setminus \mathcal{I}_R \ ;$$

for quadratic polynomials, clearly $\mathcal{B}^* = \mathcal{B}$, but for exponential maps, $\mathcal{B}^*$ is a proper subset of $\mathcal{B}$.

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[2]The parametrization in [FRS] is somewhat different, but this is of no consequence in the following.

We should remark also on the family of *unicritical* polynomials: those which have a unique critical point in $\mathbb{C}$. Such polynomials may be viewed as the topologically and combinatorially simplest polynomials of a given degree d, and they are affinely conjugate to $p_{d,c} \colon z \mapsto z^d + c$ or to $z \mapsto \lambda(1 + z/d)^d$ with $\lambda = dc^{d-1}$. These unicritical polynomials are often viewed as combinatorial interpolation between quadratic polynomials and exponential maps; see [DGH, S5, S4]. Everything we say about quadratic polynomials remains true also for unicritical polynomials, but for simplicity of exposition and notation we usually speak only of quadratic polynomials and of exponentials.

Structure of the article. In Section 2, we review the famous "MLC" conjecture for the Mandelbrot set, and then define "fibers" (introduced for the case of $\mathcal{M}$ in [S4]) of quadratic polynomials and exponential maps in parallel. This concept allows us to formulate Conjecture 8 on triviality of fibers, which is equivalent to MLC in the setting of quadratic polynomials. We also discuss a number of basic results on fibers. For ease of exposition, the theorems stated in Section 2 will be proved, separately, in Section 3.

Apart from a few somewhat subtle topological considerations, the proofs are not too difficult, but rely on a number of recent non-trivial results on the structure of exponential parameter space. We cannot comprehensively review all of these in the present article, but have attempted to present the proofs so that they can be followed without detailed knowledge of these papers.

While most of the results stated are well-known in the case of the Mandelbrot set, some observations seem to be new even in this case. (Compare, in particular, Theorem 11, which allows a simple, and formally weaker, restatement of the MLC conjecture.)

2. Local Connectivity and Trivial Fibers

Local connectivity of bifurcation loci. It was conjectured by Douady and Hubbard that the Mandelbrot set $\mathcal{M}$ is locally connected; this is perhaps *the* central open problem in holomorphic dynamics. The following is an equivalent formulation.

Conjecture 3 (MLC). *The quadratic bifurcation locus $\mathcal{B} = \partial\mathcal{M}$ is locally connected.*

One of the reasons that this conjecture is important is that it implies Conjecture 1 for quadratic polynomials: see Douady and Hubbard [DH, Exposé XXII.4] (see also [S4, Corollary 4.6], as well as Theorems 9 and 10 below).

Theorem 4 (MLC Implies Density of Hyperbolicity). *If the Mandelbrot set is locally connected, then hyperbolic dynamics is dense in the space of quadratic polynomials.*

In topological terms, the situation in exponential parameter space is very different from what we expect in the Mandelbrot set: the analog of Conjecture 3 is known to be false.

Theorem 5 (Failing Local Connectivity of Exponential Bifurcation Locus). *The exponential bifurcation locus $\mathcal{B}$ is not locally connected. More precisely, $\mathcal{B}$ is not locally connected at any point of $\mathcal{I}_R$.*

In essence, failure of local connectivity of $\mathcal{B}$ in the exponential setting was first shown by Devaney: from his proof [De] that the exponential map $\exp$ itself is not structurally stable, it also follows that $\mathcal{B}$ is not locally connected at the point $c = 0$. Theorem
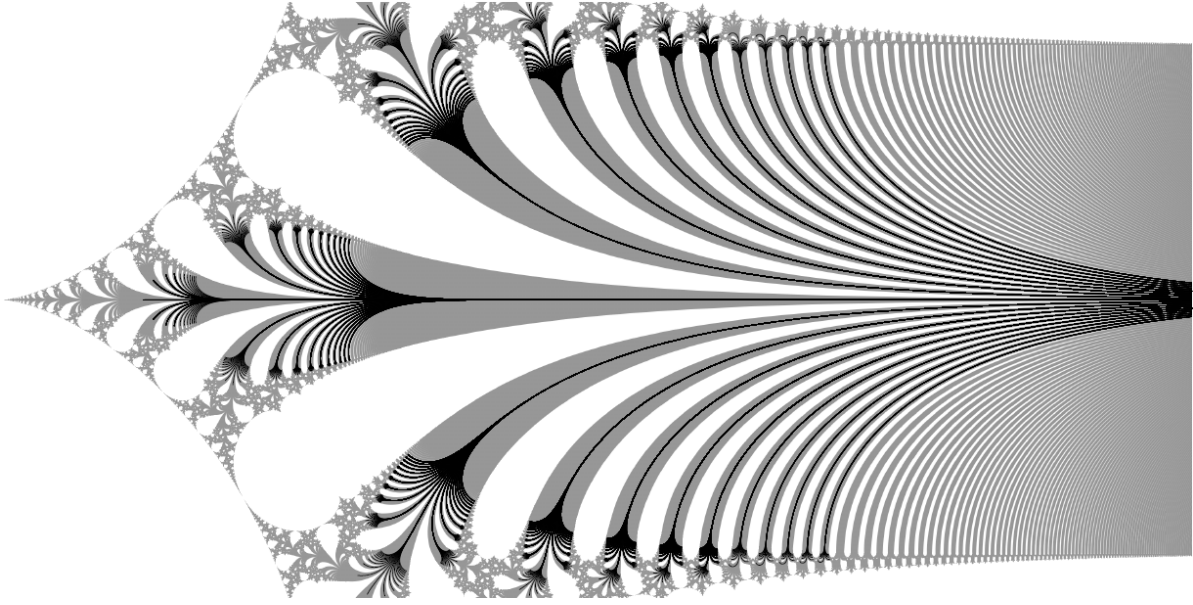
FIGURE 2. Exponential parameter space contains "Cantor bouquets", which are closed sets consisting of uncountably many disjoint simple curves. Some such curves are indicated here in black.

5 is related to the existence of so-called *Cantor bouquets* within the bifurcation locus; compare Figure 2. These consist of uncountably many curves (on parameter rays) that are locally modelled as a subset of the product of an interval and a Cantor set.

Fibers. Local connectivity of the Mandelbrot set would have a number of important consequences apart from that described by Theorem 4: for example, there are a number of topological models for $\mathcal{M}$ (such as Douady's *pinched disks* [Do] or Thurston's *quadratic minor lamination* [T]) that are homeomorphic to $\mathcal{M}$ if and only if $\mathcal{M}$ is locally connected. Moreover, MLC is equivalent to the *combinatorial rigidity conjecture*: any two maps in $\mathcal{B}$ without indifferent periodic orbits can be distinguished combinatorially (in terms of which periodic dynamic rays land together). Hence it is desirable to find another topological concept which can play the role of local connectivity in the space of exponential maps. One convenient notion of this type is provided by *triviality of fibers*, introduced for the Mandelbrot set in [S4]; it has the advantage of transferring easily to exponential parameter space. Another advantage is that even for polynomials, any possible failure of local connectivity can be made more precise by giving topological descriptions of non-trivial fibers.

Definition 6 (Separation Line and Fiber)**.** *A separation line is a Jordan arc $\gamma \subset \mathbb{C}$ in parameter space, tending to ∞ in both directions and containing only hyperbolic and finitely many parabolic parameters*[3].

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[3]Since all hyperbolic components of exponential parameter space are unbounded and ∞ is accessible, it suffices to allow just a single parabolic parameter on every separation line; for $\mathcal{M}$ at most two parabolic parameters suffice.
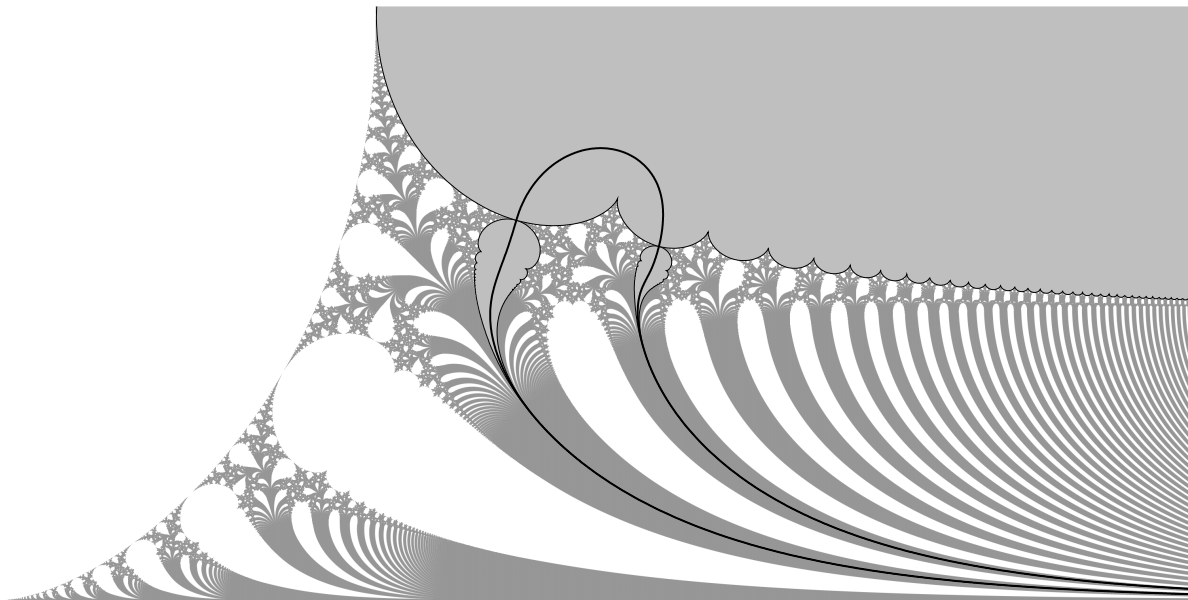
FIGURE 3. A separation line in the space of exponential maps

A separation line γ separates two points $c, c' \in \mathcal{B}^*$ if c and c' are in two different *components of $\mathbb{C} \setminus \gamma$.*

The extended fiber *of a point $c \in \mathbb{C}$ is the set of all c' which cannot be separated from c by any separation line.* The (reduced) fiber *of $c \in \mathbb{C} \setminus \mathcal{I}_R$ consists of all c' in the extended fiber of c which do not belong to $\mathcal{I}_R$.*

A fiber is called trivial *if it consists of exactly one point.*

Remark 1. This definition is somewhat different from (but equivalent to) that originally given in [S4] for quadratic polynomials. See the remark on alternative definitions of separation lines below.

Remark 2. One fundamental difference between polynomial and exponential parameter space is that the escape locus $\mathcal{I}$ is hyperbolic only in the polynomial case; for exponential parameter space, separation lines are disjoint from $\mathcal{I}$ and thus from parameter rays (an alternative definition of separation lines uses parameter rays; see below).

Remark 3. If c is a hyperbolic parameter, then it follows easily from the definition that the fiber of c is trivial. Hence all interest lies in studying non-hyperbolic fibers, and we will usually restrict our attention to this case.

Theorem 7 (Properties of fibers). *Every extended fiber $\breve{Y}$ is a closed and connected subset of parameter space. Either $\breve{Y}$ is a single point in a hyperbolic component, or $\breve{Y} \cap \mathcal{B}^* \neq \emptyset$.*

In particular, $\breve{Y}$ contains the (reduced) fiber $Y = \breve{Y} \setminus \mathcal{I}_R$; this fiber is either trivial or uncountable.

Remark 1. For fibers in the Mandelbrot set, these claims are immediate from the definition. For exponential maps, the proofs are more subtle, and require some topological considerations as well as rather detailed knowledge of parameter space.

Remark 2. Once we know that parabolic parameters have trivial fibers (which is well-known for quadratic polynomials [S4, TL, H]; for exponential maps, a proof will be provided in [R4]), it also follows that fibers are pairwise disjoint.

Armed with the definition of fibers, we can now propose the following central conjecture.

Conjecture 8 (Fibers are trivial). *For the spaces of quadratic polynomials and exponential maps, all fibers are trivial.*

Since any non-hyperbolic stable component would be contained in a single fiber, it follows immediately that triviality of fibers would settle density of hyperbolicity:

Theorem 9 (Triviality of fibers implies density of hyperbolicity). *Conjecture 8 implies Conjecture 1.*   ∎

Triviality of fibers and local connectivity of the Mandelbrot set. Following Milnor [M, Remark after Lemma 17.13], we say that a topological space X is *locally connected at a point* $x \in X$ if x has a neighborhood base consisting of connected sets.[4] Triviality of the fiber of a parameter c in the quadratic bifurcation locus implies local connectivity of the Mandelbrot set at c. In fact, all known proofs of local connectivity of $\mathcal{M}$ at given parameters do so by actually establishing triviality of fibers.

The converse question — in how far local connectivity implies triviality of fibers — is more subtle. For example, a full, compact, connected set K may well contain non-accessible points at which K is locally connected; see Figure 4(a). On the other hand, triviality of the fiber of a parameter c implies that there is a parameter ray landing at c (see Theorem 11 below), and hence that c is accessible from the complement of $\mathcal{M}$. So local connectivity at c does not formally imply triviality of the fiber of c, but there is the following, more subtle, connection.

Theorem 10 (Trivial Fibers and MLC). *Let c belong to the quadratic bifurcation locus $\mathcal{B}$. Then the following are equivalent:*

(a) *the Mandelbrot set $\mathcal{M}$ is locally connected at every point of the fiber of c;*
(b) *the fiber of c is trivial.*

In particular, local connectivity of the Mandelbrot set is equivalent to triviality of all fibers in the quadratic bifurcation locus.

Triviality of Fibers and Landing of Parameter Rays. For the Mandelbrot set, by Carathéodory's theorem [M, Theorem 17.14] local connectivity implies that every parameter ray lands, and the map assigning to every external angle the landing point of the corresponding parameter ray is a continuous surjection $\mathbb{S}^1 \to \mathcal{B}$.

Again, replacing local connectivity by triviality of fibers allows us to obtain a statement regarding the landing of rays which is true in both quadratic and exponential parameter space.

Theorem 11 (Landing of rays implies triviality of fibers). *Let $c \in \mathcal{B}^*$. Then the following are equivalent.*

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[4]Sometimes this property is instead referred to as "connected im kleinen" (cik) at x, and the term "locally connected at x" is reserved for what Milnor calls "openly locally connected at x".
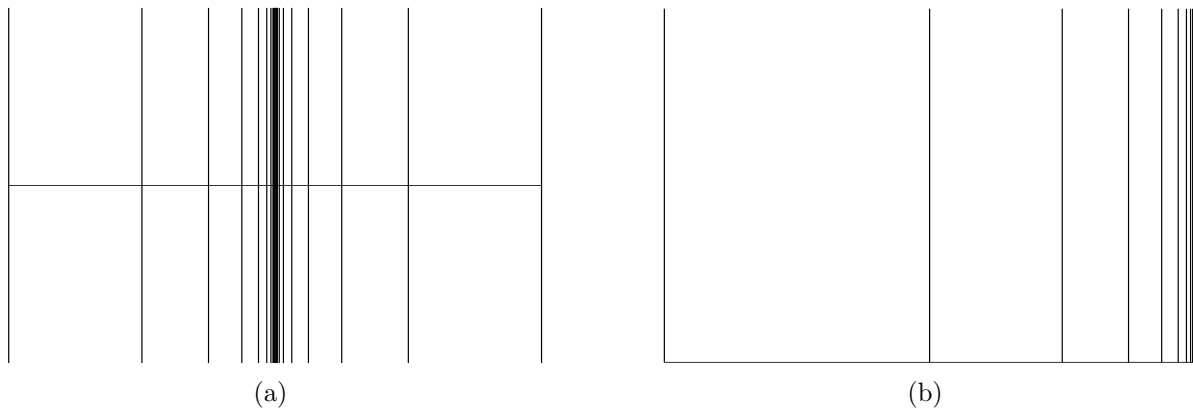
FIGURE 4. Examples of compact connected sets $K \subset \mathbb{C}$ with connected complement such that (a) K is locally connected at a point $z_0 \in K$, but z_0 is not accessible from $\mathbb{C} \setminus K$ (where z_0 is the point at the center of the figure); (b) every $z \in K$ is accessible from $\mathbb{C} \setminus K$ (and hence the landing point of an external ray), but K is not locally connected.

(a) *The fiber of c is trivial.*
(b) *Every point in the fiber of c is the landing point of a parameter ray.*

In particular, triviality of all fibers is equivalent to the fact that every point of $\mathcal{B}^*$ is the landing point of a parameter ray.

The last sentence in the theorem provides a convenient way of stating Conjecture 8 without the definition of fibers.

The previous result leads to an interesting observation about the Mandelbrot set, which is new as far as we know: *MLC is equivalent to the claim that every $c \in \partial \mathcal{M}$ is accessible from $\mathbb{C} \setminus \mathcal{M}$.* For general compact sets, this is far from true (compare Figure 4(b)).

In this context, there is a difference between the situation for the Mandelbrot set and that of exponential parameter space. In the former case, triviality of the fiber of c implies that *all* parameter rays accumulating on c land. In the latter case, we can only conclude that *one* of these rays lands. The problem is that escaping parameters are contained in the bifurcation locus; it is compatible with our current knowledge that one parameter ray might accumulate on another ray together with its landing point (compare Figure 5), in which case the corresponding fiber could still be trivial. This might lead us to formulate a stronger variant of Conjecture 8: *all fibers are trivial, and furthermore all parameter rays land.* It seems plausible that these conjectures are equivalent (we believe that both are true).

Alternative definitions of separation lines. We defined separation lines as curves through hyperbolic components. We did so since this gives a simple definition and does not require the landing of periodic parameter rays in the exponential family (a fact that was proved in [S1], which is however not formally published).

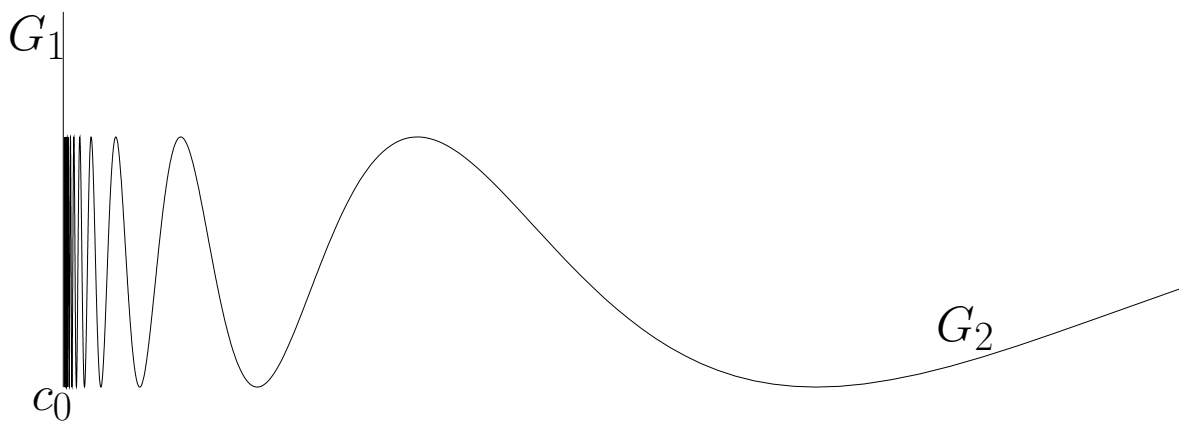There are a number of other definitions we could have chosen; for example,

FIGURE 5. It is conceivable that an extended fiber in exponential parameter space consists of one ray G_1 landing at some parameter c_0 together with a second ray G_2 which accumulates not only on c_0, but also on a segment of G_1. In this case, the fiber of c_0 would be trivial, but the ray G_2 clearly does not land at c_0.

(a) A *separation line* is a curve consisting either of two parameter rays landing at a common parabolic parameter, or of two parameter rays landing at distinct parabolic parameters, together with a curve which connects these two landing points within a single hyperbolic component.

(b) A *separation line* is a curve as in (a), except that we also allow two parameter rays landing at a common parameter for which the singular value is preperiodic.

(c) A *separation line* is a Jordan arc, tending to infinity in both directions, containing only finitely many parameters which are not escaping or hyperbolic.

These alternative definitions will yield a theory of fibers for which all of the above results remain true. However, it is a priori conceivable that separation lines run through parameters with nontrivial fibers, in which case the fiber of a nearby parameter $c \in \mathcal{B}^*$ may depend on the definition of separation lines[5]. On the other hand, the question of *triviality* of such a fiber, and hence Conjecture 8, is independent of this definition.

3. PROOFS

We begin by showing that the exponential bifurcation locus is not locally connected.

Proof of Theorem 5. (Exponential bifurcation locus not locally connected.) As already remarked, failure of local connectivity in the exponential setting follows already from Devaney's proof of structural instability of exp [De]. Essentially, he showed that the interval $[0, \infty) \subset \mathcal{B}$ is accumulated on by curves in hyperbolic components (compare Figure 6).

We will now indicate how to prove failure of local connectivity at every point of $\mathcal{I}_R$ using a similar idea, together with more detailed knowledge of exponential parameter

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[5]The original definition in [S4] for $\mathcal{M}$ is (b), which is shown to be equivalent to (a); for $\mathcal{M}$, all parabolic parameters have trivial fibers.
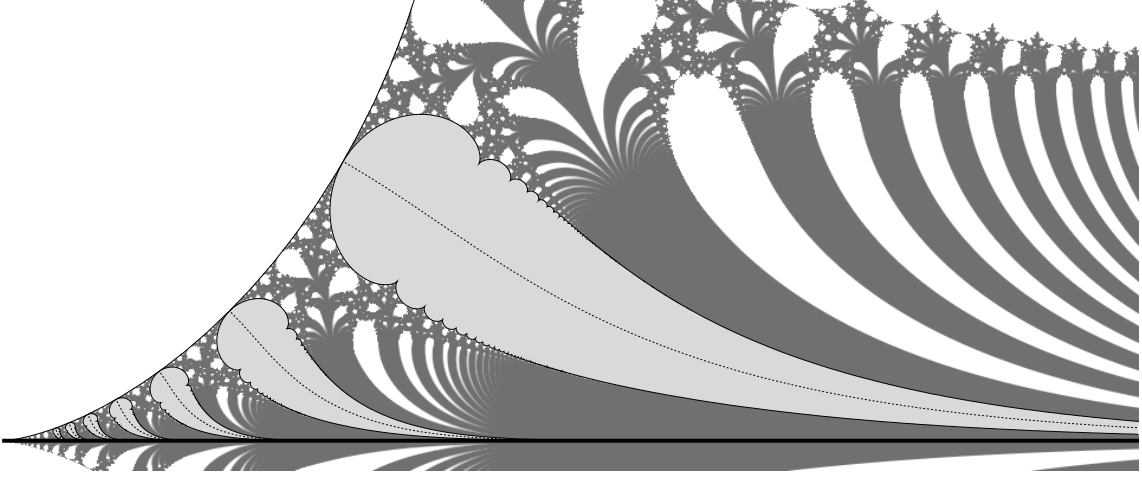
FIGURE 6. Curves in hyperbolic components accumulating on the interval $[-1, \infty)$ in exponential parameter space.

space. More precisely, let $G_{\underline{s}} \colon (0, \infty) \to \mathbb{C}$ be a parameter ray in exponential parameter space. We claim that, for every $t_0 > 0$, the curve $G_{\underline{s}} \colon [t_0, \infty) \to \mathbb{C}$ is the uniform limit of curves $\gamma_n \colon [t_0, \infty) \to \mathbb{C}$ within hyperbolic components W_n of period n.

Every small neighborhood U of a parameter $G_{\underline{s}}(t)$ with $t > t_0$ then intersects the boundaries of infinitely many of these components, and these boundaries are separated from $G_{\underline{s}}(t)$ within U by the curves γ_n. This proves the theorem.

In the case where t_0 is sufficiently large, the existence of the curves γ_n is worked out in detail in [RS1, Section 4]. Here, we will content ourselves with indicating the overall structure of the proof, for arbitrary $t_0 > 0$.

Let $c_0 = G_{\underline{s}}(t)$ for $t \geq t_0$, and let $z_n := E^n(c_0) := E_{c_0}^{\circ n}(c_0)$ denote the singular orbit of E_{c_0}. Since c_0 is on a parameter ray, the real parts of the z_n converge to infinity like orbits under the (real) exponential function. More precisely, we have the asymptotics

$$z_n = F^{\circ n}(T) + 2\pi i s_{n+1} + o(1),$$

where $F(x) = \exp(x) - 1$, $\underline{s} = s_1 s_2 s_3 \dots$ and T is some positive real number. Given the expansion of exp in the right half plane, it should not be surprising that

$$\left| (E^n)'(c_0) \right| \to \infty$$

as $n \to \infty$ (for a proof, see [BBS, Lemma 6]). Furthermore, this growth of the derivative is uniform for $t \geq t_0$.

It follows readily (compare [BBS, Lemma 3]) that, for sufficiently large n, $c_0 = G_{\underline{s}}(t)$ can be perturbed to a point $c_n = \gamma_n(t)$ whose singular orbit follows that of E_c closely until the $(n-2)$th iteration, where the two orbits differ by $i\pi$. In other words,

$$\left| E^k(c_n) - z_k \right| \ll 1 \quad \text{for all } k = 0, 1, \dots, n-3,$$

while

(*) $$E^{n-2}(c_n) = z_{n-2} + i\pi.$$

This implies that the real part of $E_{c_n}^{\circ n-1}(c_n)$ is very negative, and hence $E_{c_n}^{\circ n}(c_n)$ is very close to c_0. So the orbit of the singular value c_n is "almost" periodic of period n.

The contraction along this orbit is such that a certain disk around c_n is mapped into itself, and E_{c_n} has an attracting periodic orbit of period n (similarly as in [BR, Lemma 7.1]; compare [S2, Lemma 3.4]). It is easy to see that $c_n = \gamma_n(t)$ is continuous in t and converges uniformly to $G_{\underline{s}}(t)$ for $n \to \infty$, as desired. ∎

Remark 1. We remark that the curves γ_n constructed in the proof converge to $G_{\underline{s}}\big([t_0, \infty)\big)$ "from above", in the sense that they tend to infinity in the unique component of

$$\{\operatorname{Re} z > \operatorname{Re} G_{\underline{s}}(t_0)\} \setminus G_{\underline{s}}\big([t_0, \infty)\big)$$

which contains points with arbitrarily large imaginary parts. In equation (*), we could have just as well chosen c_n such that $E^{n-2}(c_n) = z_{n-2} - i\pi$; in this case the curves γ_n would converge to $G_{\underline{s}}\big([t_0, \infty)\big)$ from below. We will use this fact in the proof of Lemma 13 below.

Remark 2. We do not currently know anything about local connectivity of $\mathcal{B}$ at points of $\mathcal{B}^*$. It seems conceivable that $\mathcal{B}$ is locally connected exactly at the points of $\mathcal{B}^*$; but in our view the more relevant question is whether all fibers of points in $\mathcal{B}^*$ are trivial.

To begin our discussion of fibers, we note the following elementary consequences of their definition.

Lemma 12 (Extended Fibers)**.** *Every extended fiber $\check{Y}$ is a closed subset of $\mathbb{C}$; the closure $\hat{Y}$ of $\check{Y}$ in the Riemann sphere is compact and connected.*

(In the space of quadratic polynomials, every extended fiber is bounded and hence $\hat{Y} = \check{Y}$; in the space of exponential maps, every non-hyperbolic extended fiber is unbounded and hence has $\hat{Y} = \check{Y} \cup \{\infty\}$.)

Proof. Let $\check{Y}$ be the extended fiber of a point c_0. The set of points not separated from c_0 by a given separation line is a closed subset of $\mathbb{C}$. So extended fibers are defined as an intersection of a collection of closed subsets of $\mathbb{C}$, and therefore closed themselves.

It is easy to verify that a point c which is separated from c_0 by a collection of separation lines can also be separated from c by a single separation line. Furthermore, one can verify that only countably many separation lines γ are necessary to separate c_0 from all points outside $\check{Y}$. For example, we can require that the intersection of γ with any hyperbolic component W is a hyperbolic geodesic of W. Since the set of parabolic parameters is countable and every separation line runs through hyperbolic components and finitely many parabolic parameters, there are only countably many such separation lines.

It follows that $\check{Y}$ can be written as a nested countable intersection of closed, connected subsets of the plane; this proves connectivity. ∎

Before we prove the remaining theorems, we require some preliminary combinatorial and topological considerations in exponential parameter space. These become necessary mainly because we need to take into account the possibility of parameter rays accumulating on points of $\mathcal{I}_R$.

We begin by noting that there is a natural combinatorial compactification of exponential parameter space, as follows. The external addresses considered so far are elements of $\mathbb{Z}^{\mathbb{N}}$; we will sometimes call these *infinite* external addresses. We also introduce *intermediate* external addresses: these have the form $s_1 \ldots s_{n-1}\infty$, where $n \geq 1$, $s_1, \ldots, s_{n-2} \in \mathbb{Z}$

and $s_{n-1} \in \mathbb{Z} + 1/2$; note that there is a unique intermediate external address of length $n = 1$, namely ∞. The set of all infinite and intermediate external addresses will be denoted $\overline{\mathcal{S}}$. The lexicographic order induces a complete total order on $\overline{\mathcal{S}} \setminus \{\infty\}$ and a complete cyclic order on $\overline{\mathcal{S}}$; compare [RS2, Section 2].

We can then define a natural topology on $\tilde{\mathbb{C}} = \mathbb{C} \cup \overline{\mathcal{S}}$, which has the property that $G_{\underline{s}}(t) \to \underline{s}$ in this topology as $t \to +\infty$. The space $\tilde{\mathbb{C}}$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$, where $\mathbb{C}$ corresponds to the interior of the disk, and $\overline{\mathcal{S}}$ to the unit circle. (This construction is analogous to compactifying parameter space of quadratic polynomials by adding a circle of external angles at infinity.) See also [RS2, Appendix A].

For any parameter ray $G_{\underline{s}}$, we now set $G_{\underline{s}}(\infty) := \underline{s} \in \tilde{\mathbb{C}}$, giving a parametrization $G_{\underline{s}} : (0, \infty] \to \tilde{\mathbb{C}}$. Similarly, recall the curves γ_n accumulating on the parameter ray $G_{\underline{s}}$ as constructed in the proof of Theorem 5. They can be extended continuously by setting $\gamma_n(\infty) = \underline{s}_n$, where $\underline{s}_n$ is an intermediate external address and $\underline{s}_n \to \underline{s}$. Armed with this terminology, we can state and prove the following key fact, which we will use repeatedly.

Lemma 13 (Accumulation on parameter rays). *Let $A \subset \tilde{\mathbb{C}}$ be connected, and let $\tilde{A}$ be the closure of A in $\tilde{\mathbb{C}}$. Suppose that A intersects at most finitely many hyperbolic components and that $A \cap \overline{\mathcal{S}}$ contains at most finitely many intermediate external addresses.*

Suppose that G is a parameter ray, and that $G(t_0) \in \tilde{A}$ for some $t_0 \in (0, \infty]$. Then either $\tilde{A} \subset G([t, \infty])$ for some $0 < t \le t_0$, or $G((0, t_0]) \subset \tilde{A}$.

Proof. If $G((0, t_0]) \subset \tilde{A}$, then nothing is left to prove, so we may assume that there is a $t \in (0, t_0)$ with $G(t) \notin \tilde{A}$ and then show that $\tilde{A} \subset G([t, \infty])$. There is some $\varepsilon > 0$ with $\mathbb{D}_\varepsilon(G(t)) \cap A = \emptyset$. Recall from the proof of Theorem 5 and the subsequent remark that $G([t, \infty))$ is accumulated on from above resp. below by curves $\gamma_n^+ : [t, \infty) \to \mathbb{C}$ and $\gamma_n^- : [t, \infty) \to \mathbb{C}$, each contained in a hyperbolic component of period n. Also recall that we can extend these curves continuously by setting $\gamma_n^\pm := \underline{s}_n^\pm$ with suitable intermediate external addresses $\underline{s}_n^\pm$. We then have $\underline{s}_n^+ \searrow \underline{s}$ and $\underline{s}_n^- \nearrow \underline{s}$.

By assumption, A intersects at most finitely many of the curves $\gamma_n^\pm$. So for sufficiently large n, A is disjoint from

$$K_n := \overline{\mathbb{D}_\varepsilon(G(t))} \cup \gamma_n^+([t, \infty]) \cup \gamma_n^-([t, \infty]) .$$

Let U_n be the component of $\tilde{\mathbb{C}} \setminus K_n$ containing $G(T)$ for sufficiently large T. If ε is sufficiently small, then $G(t_0) \in U_n$, hence $A \subset U_n$ and so

$$A \subset \bigcap_n U_n \subset G_{\underline{s}}([t, \infty))$$

as desired. (Compare [R2, Corollary 11.4] for a similar proof, using parameter rays accumulating on $G_{\underline{s}}$ instead of the curves $\gamma_n^\pm$; recall Figure 2.) ∎

Lemma 13 can be applied, in particular, to the accumulation sets of parameter rays, as stated in the following corollary.

Corollary 14 (Accumulation sets of parameter rays). *Let $G_{\underline{s}}$ be a parameter ray, and let L denote the accumulation set of $G_{\underline{s}}$ in $\tilde{\mathbb{C}}$. If G is a parameter ray and $G(t_0) \in L$ for some $t_0 \in (0, \infty]$, then $G((0, t_0]) \subset L$.*

(In particular, the accumulation set of G is contained in L.)

Proof. Apply the previous lemma to the sets $A = A_t = G_{\underline{s}}\big((0,t]\big)$ with $t > 0$. Since A contains points at arbitrarily small potentials, and since parameter rays are pairwise disjoint, the first alternative in the conclusion of the lemma cannot hold. Hence (writing $\widetilde{A_t}$ for the closure of A_t in $\tilde{\mathbb{C}}$ as before) we have

$$G\big((0,t_0]\big) \subset \bigcap_{t>0} \widetilde{A_t} = L. \qquad \blacksquare$$

We also require the following fundamental theorem from [RS1] about exponential parameter space.

Theorem 15 (The Squeezing Lemma (connected sets version)). *Let A be an unbounded connected subset of exponential parameter space which contains only finitely many indifferent and no hyperbolic parameters, and let $\tilde{A}$ be the closure of A in $\tilde{\mathbb{C}}$. Then every $\underline{s} \in \widetilde{A} \cap \overline{\mathcal{S}}$ is an infinite external address for which a parameter ray $G_{\underline{s}}$ exists, and $G_{\underline{s}}(t) \in \widetilde{A}$ for all sufficiently large t.*

Remark 1. In [RS1, Theorem 1.3], the Squeezing Lemma is formulated for curves rather than connected sets. The proof given there also proves the above version; compare the sketch below. The name of the result comes from the "squeezing" around a parameter ray by nearby hyperbolic components (or parameter rays) as in the proof of Lemma 13.

Remark 2. The idea of the proof of the Squeezing Lemma goes back to the original proof [S1, Theorem V.6.5], [S3] that the boundary of every hyperbolic component in exponential parameter space is a connected subset of the plane. A suitable formulation of the Squeezing Lemma can also be used to prove this fact; see [RS1, Theorem 1.2].

Sketch of proof. The proof requires a number of technical and combinatorial considerations. We will describe the underlying strategy and refer the reader to [RS1] for details.

Let $\underline{s} \in \widetilde{A} \cap \overline{\mathcal{S}}$. First, let us show that $\underline{s}$ cannot be an intermediate external address. Indeed, otherwise there is a unique hyperbolic component W which is associated to $\underline{s}$ in the following sense [S2]: if $\gamma : [0,\infty) \to W$ is a curve along which the multiplier of the attracting orbit tends to zero, then $\underline{s}$ is the sole accumulation point of γ in W. Since $A \cap W$ is empty, we can draw a separation line through W and two child components (one slightly above W and one slightly below) which separates A from the address $\underline{s}$. More details can be found in [RS1, Section 6].

So $\underline{s}$ is an infinite external address; i.e. $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$. If there is no parameter ray associated to $\underline{s}$, then this means [FS] that at least some of the entries in the sequence $\underline{s}$ grow extremely fast. Every entry in $\underline{s}$ which exceeds all previous ones implies the existence of a separation line in $\tilde{\mathbb{C}}$ which encloses parameter rays at external addresses near $\underline{s}$, and the domains thus enclosed shrink to $\underline{s} \in \tilde{\mathbb{C}}$. Hence some of these separation lines separate A from external addresses near $\underline{s}$, a contradiction. (The detailed proof uses the combinatorial structure of *internal addresses* and can be found in [RS1, Section 7].) It follows that the address $\underline{s}$ must indeed have a parameter ray associated to it.

We can hence apply Lemma 13 with $t_0 = \infty$. It follows that $G_{\underline{s}}(t) \in \tilde{A}$ for all sufficiently large t, as claimed. $\qquad \blacksquare$

We show below that every parameter ray has an accumulation point in $\mathcal{B}^*$. For now, we only note the following.

Lemma 16. *Every parameter ray has an accumulation point in $\mathcal{B}$. (That is, a parameter ray cannot land at infinity.)*

Proof. For the Mandelbrot set, this is clear. For exponential parameter space, it follows by applying the Squeezing Lemma to the connected set $A := G_{\underline{s}}\big((0,1]\big)$. Indeed, if A is bounded, then $G_{\underline{s}}$ has at least one finite accumulation point, and there is nothing to prove. Otherwise, let $\tilde{A}$ be the closure of A in $\tilde{\mathbb{C}}$. Then there exists an infinite external address $\underline{s}' \in \tilde{A} \cap \overline{\mathcal{S}}$ so that $G_{\underline{s}'}(t) \in \tilde{A}$ for all sufficiently large t. Since A does not contain parameters on parameter rays at arbitrarily high potentials, it follows that $G_{\underline{s}'}(t)$ belongs to $\overline{A} \setminus A$, and hence to the accumulation set of $G_{\underline{s}}$. ∎

The following is essentially a weak version of Douady and Hubbard's "branch theorem" [DH, Proposition XXII.3]; see also [S4, Theorem 3.1].

Lemma 17. *Let $\breve{Y}$ be a non-hyperbolic extended fiber. Then $\breve{Y}$ contains the accumulation set of at least one but at most finitely many parameter rays.*

Proof. For quadratic polynomials, the fact that every fiber contains the accumulation set of a parameter ray is immediate. Indeed, if $c_0 \in \partial\mathcal{M}$, then c_0 is contained in some *prime end impression* (compare e.g. [M, Chapter 17] or [P, Chapter 2] for background on the theory of prime ends). This impression contains the accumulation set of an associated parameter ray (this set is called the set of *principal points* of the prime end). Such an accumulation set clearly cannot be separated from c_0 by any separation line.

That there are only finitely many such parameter rays follows from the usual Branch Theorem, see [DH, Exposé XXII] or [S4, Theorem 3.1], which states that the Mandelbrot set branches only at hyperbolic components and at postsingularly finite parameters.

For exponential maps, the finiteness statement follows from the corresponding fact for Multibrot sets. Indeed, suppose we had infinitely many parameter rays $G_{\underline{s}_1}, G_{\underline{s}_2}, \ldots$ which are not separated from one another by any separation line. Then it follows by combinatorial considerations that all $\underline{s}_j$ are bounded sequences, and they differ from one another at most by 1 in each entry. (The argument is similar as in the proof of the Squeezing Lemma: if two addresses differ by more than 1 in one entry, then the "internal address algorithm" generates a separation line separating the two. Similarly, if $\underline{s}$ was unbounded, then for every bounded external address $\underline{t}$ there is a separation line which surrounds $\underline{s}$ — i.e., separates $\underline{s}$ from ∞ in $\tilde{\mathbb{C}}$ — and separates $\underline{s}$ from $\underline{t}$. But any separation line which surrounds $\underline{s}$ must also surround all other $\underline{s}_j$, which contradicts the fact that there are bounded external addresses between any two elements of $\overline{\mathcal{S}}$.)

So the entries of the addresses $\underline{s}_j$ are uniformly bounded, and for every Multibrot set $\mathcal{M}_d$ of sufficiently high degree d, there are parameter rays $G_{\underline{s}_1}, G_{\underline{s}_2}, \ldots$ at corresponding angles. The Branch Theorem for Multibrot sets [S4, Theorem 3.1 and Corollary 8.5] shows that some of the parameter rays $G_{\underline{s}_i}$ are separated from each other by a separation line for $\mathcal{M}_d$. But then it follows combinatorially that there is a similar separation line in exponential parameter space, in contradiction to our assumption. Compare [RS2, Theorem A.3].

To see that there is at least one parameter ray contained in every non-hyperbolic extended fiber $\breve{Y}$, recall from Lemma 12 that $\breve{Y} \cup \{\infty\}$ is a compact and connected subset of the Riemann sphere. So every component of $\breve{Y}$ is unbounded; the Squeezing

Lemma implies that each such component contains points on some parameter ray $G_{\underline{s}}$. But since separation lines cannot separate the ray $G_{\underline{s}}$, it follows that $\breve{Y}$ contains the entire ray $G_{\underline{s}}$ and hence also its accumulation set. ∎

Lemma 18 (Extended fibers are connected). *Let $\breve{Y}$ be an extended fiber. Then $\breve{Y}$ and $\breve{Y}_{\mathcal{B}} := \breve{Y} \cap \mathcal{B}$ are connected subsets of $\mathbb{C}$.*

Proof. We can assume that $\breve{Y}$ is not a hyperbolic fiber, as otherwise the claim is trivial.

For quadratic polynomials $\breve{Y}$ is a compact and connected subset of $\mathbb{C}$ (Lemma 12). Furthermore, every component of $\breve{Y} \setminus \mathcal{B}$ (if any) would be a (non-hyperbolic) stable component of the Mandelbrot set. Since all such components are simply connected, removing them from $\breve{Y}$ does not disconnect $\breve{Y} \cap \mathcal{B}$.

It remains to deal with the exponential case. Let $G_1, \ldots, G_n$ be the parameter rays intersecting $\breve{Y}$, with external addresses $\underline{s}_1, \ldots, \underline{s}_n$. Then all G_i are contained in $\breve{Y}$. Let $\tilde{Y}$ be the closure of $\breve{Y}$ in $\tilde{\mathbb{C}}$; then, by the Squeezing Lemma, $\tilde{Y} = \breve{Y} \cup \{\underline{s}_1, \ldots, \underline{s}_n\}$. Note that $\tilde{Y}$ is a compact connected subset of $\tilde{\mathbb{C}}$ (as in Lemma 12, it can be written as a countable nested intersection of compact connected subsets of $\tilde{\mathbb{C}}$).

As in the quadratic setting, any stable component U of exponential maps is simply connected (since boundaries of hyperbolic components and parameter rays, both of which are unbounded, are dense in $\mathcal{B}$). Hence $\tilde{Y}_{\mathcal{B}} := \breve{Y}_{\mathcal{B}} \cup \{\underline{s}_1, \ldots, \underline{s}_n\}$ is also compact and connected.

Now suppose by contradiction that $\breve{Y}_{\mathcal{B}} = A_0 \cup A_1$, where A_0 and A_1 are nonempty, closed and disjoint. Let $\tilde{A}_j$ be the closure of A_j in $\tilde{\mathbb{C}}$. Since $\tilde{Y}_{\mathcal{B}} = \tilde{A}_0 \cup \tilde{A}_1$ is connected, we must have $\tilde{A}_0 \cap \tilde{A}_1 \neq \emptyset$. I.e., some $\underline{s}_j$ belongs to both $\tilde{A}_0$ and $\tilde{A}_1$; let C be the component of $\tilde{A}_0$ containing $\underline{s}_j$.

By the boundary bumping theorem [Na, Theorem 5.6], every component of $\tilde{A}_0$ contains one of the addresses $\underline{s}_i$, so $\tilde{A}_0$ has only finitely many connected components. This implies that $C \supsetneq \{\underline{s}_j\}$. By Lemma 13, it follows that $C \cap G_j \neq \emptyset$. Since G_j is connected, in fact $G_j \subset A_0$. Likewise, $G_j \subset A_1$, which contradicts the assumption that $A_0 \cap A_1 = \emptyset$.

So $\breve{Y}_{\mathcal{B}}$ is connected, as is $\breve{Y}$ itself. ∎

Proof of Theorem 2. (Bifurcation locus is connected.) Suppose that the bifurcation locus $\mathcal{B}$ is not connected. Then there is some stable component U such that two components C_1 and C_2 of ∂U belong to different connected components of $\mathbb{C} \setminus U$. Since boundaries of hyperbolic components are connected subsets of $\mathbb{C}$ [S3, RS1] (recall Remark 2 after Theorem 15), U must be a non-hyperbolic stable component, and hence $\overline{U}$ is contained in a single extended fiber $\breve{Y}$ (no separation line can separate any two points in $\overline{U}$). However, then C_1 and C_2 would belong to different components of $\breve{Y} \setminus U$, which contradicts Lemma 18. ∎

Remark. Theorem 2 can also be proved directly from the "Squeezing Lemma"; see [RS1, Proof of Theorem 1.1].

We have now proved a number of results regarding *extended* fibers. Fibers themselves can be more difficult to deal with in the exponential setting, because they may (at least *a priori*) not be closed. For example, from what we have shown so far, it is conceivable

that an extended fiber is completely contained in $\mathcal{I}_R$, and hence does not contain a (reduced) fiber. We shall now show that this is not the case, and that any extended fiber in fact intersects $\mathcal{B}^*$ in either one or uncountably many points.

Theorem 19 (Accumulation sets of parameter rays). *Every parameter ray has at least one accumulation point in $\mathcal{B}^*$; if there is more than one such point, there are in fact uncountably many.*

Furthermore, let Y be a fiber. Then either

- *Y is trivial (i.e., consists of exactly one point); if Y is non-hyperbolic, then there is at least one parameter ray landing at the unique point of Y; or*
- *Y is not trivial, in which case $Y \cap \mathcal{B}^*$ is uncountable.*

Proof. For the quadratic family, the accumulation set of every parameter ray is contained in the boundary of the Mandelbrot set, which equals $\mathcal{B}^*$. Any fiber Y is a closed, connected subset of $\mathcal{M}$, and hence either consists of a single point or has the cardinality of the continuum. If the fiber has interior, then its boundary is contained in $\mathcal{B}^* \cap Y$ and also has the cardinality of the continuum. If the fiber is trivial, then clearly *every* parameter ray accumulating on Y must land at the single point of Y.

So let us now consider the case of the exponential bifurcation locus. Let $G_{\underline{s}}$ be a parameter ray, and let $\breve{Y}$ be the extended fiber containing $G_{\underline{s}}$. By Lemma 17, $\breve{Y}$ contains at most finitely many parameter rays $G_1, \ldots, G_n$, at addresses $\{\underline{s}_1, \ldots, \underline{s}_n\}$. (Our original ray $G_{\underline{s}}$ will be one of these.) Denote by L_i the set of all accumulation points of G_i (as $t \to 0$) in $\tilde{\mathbb{C}}$.

Note that, if $L_i \cap \mathcal{I}_R \neq \emptyset$, say $G_j(t) \in L_i$ for some $j \in \{1, \ldots, n\}$ and $t \in (0, \infty]$, then $G_j\big((0,t]\big) \subset L_i$ and $L_j \subset L_i$ by Corollary 14.

Claim. *Let $L \subset (\breve{Y} \cap \mathcal{B}) \cup \{\underline{s}_1, \ldots \underline{s}_n\} \subset \tilde{\mathbb{C}}$ be nonempty, compact and connected, and suppose that there are no i and $t > 0$ with $L \subset G_i\big([t, \infty]\big)$.*

Then $L \cap \mathcal{B}^*$ is either uncountable or a singleton. In the latter case,

(a) *if $L \not\subset \mathcal{B}^*$, then there is j such that $G_j \cap L \neq \emptyset$ and G_j lands at the unique point of $L \cap \mathcal{B}^*$;*

(b) *every connected component of $\mathbb{C} \setminus L$ contains infinitely many parameter rays, and hence is not contained in $\breve{Y}$.*

Note that, in (a), we do not claim that *all* rays which intersect L land at the unique point of $L \cap \mathcal{B}^*$. To illustrate the claim in this case, it may be useful to imagine L to be the set from Figure 5. Other model cases to imagine are those where L is the union of of finitely many parameter rays landing at a common point, or where L is an indecomposable continuum with one or more parameter rays dense in L (as e.g. in the standard Knaster (or "buckethandle") continuum [K, §43, V, Example 1]).

Using the claim, we can prove both statements of the theorem. For the first part, we let L be the accumulation set of $G_{\underline{s}}$; by Corollary 14, we do not have $L \subset G_s\big([t, \infty)\big)$ for any i and $t > 0$, so the claim applies. For the second part, set $L := (\breve{Y} \cap \mathcal{B}) \cup \{\underline{s}_1, \ldots, \underline{s}_n\}$. (The final part of the claim implies, in particular, that if $L \cap \mathcal{B}^* = Y \cap \mathcal{B}^*$ is a singleton, then $\breve{Y}$ has no interior components, and hence Y is trivial, as claimed.)

Proof of the Claim. Note that it follows from Lemma 13 and the assumption that, for every $j \in \{1, \ldots, n\}$, there is some $T \in [0, \infty]$ such that $L \cap G_j = G_j\big((0, T]\big)$ (where we understand the interval $(0, T]$ to be empty in the case of $T = 0$).

We will proceed by removing isolated end pieces of parameter rays from L. More precisely, suppose that there is some $j \in \{1, \ldots, n\}$ and some $t > 0$ such that $G_j\big((t, \infty]\big) \cap L$ is a relatively open subset of L. Choose $t_0 \geq 0$ minimal such that all $t > t_0$ have this property.

Then $G_j\big((t_0, \infty]\big)$ is a relatively open subset of L, which we will call an "isolated end piece". Note that the relative boundary of this piece in L is either the singleton $G_j(t_0)$, if $t_0 > 0$, or the accumulation set L_j of the ray G_j otherwise; so in either case this boundary is connected. By the boundary bumping theorem, $L' := L \setminus G_j\big((t_0, \infty]\big)$ is a compact connected subset of $\tilde{\mathbb{C}}$. Furthermore, L_j does not contain any point $G_j(t)$ with $t > t_0$ by choice of t_0. So $L_j \subset L'$, and hence L' is nonempty and satisfies the assumptions of the claim. Note that $L' \cap \mathcal{B}^* = L \cap \mathcal{B}^*$.

We can apply this observation repeatedly to remove such isolated end pieces of parameter rays from L. Note that, if $t_0 = 0$, then it could be that L' contains an isolated end piece of a parameter ray which was not isolated in L. (Recall Figure 5.) However, since this happens at most n times, in finitely many steps we obtain a set $L_0 \subset L$, satisfying the assumptions of the claim and with $L_0 \cap \mathcal{B}^* = L \cap \mathcal{B}^*$, such that furthermore

(*) if $t > 0$ is such that $G_j\big((t, \infty]\big) \cap L_0 \neq \emptyset$, then $G_j\big((t, \infty]\big) \cap L_0$ is not relatively open in L_0.

We observe that (*) implies the following stronger property:

(**) if $G_j(t_1) \in L_0$, then $G_j\big([t, t_1]\big)$ is a nowhere dense subset of L_0 for all $t \in (0, t_1)$.

Equivalently, there are no $t \in (0, t_1)$ and $\varepsilon < \min(|t|, |t_1 - t|)$ such that $I := G_j((t - \varepsilon, t + \varepsilon))$ is relatively open in L_0. To prove (**), suppose by contradiction that such t and ε exist. By the boundary bumping theorem, every connected component of $L_0 \setminus I$ must contain one of the two endpoints of I. Hence there are at most two such components, and these are therefore both open and closed in $L_0 \setminus I$. Furthermore, by Lemma 13, any component of $L_0 \setminus I$ which intersects $G_j\big([t + \varepsilon, \infty]\big)$ is contained in $G_j\big([t + \varepsilon, \infty]\big)$. Together, these facts imply that $G_j\big((t + \varepsilon, \infty]\big) \cap L_0$ is a relatively open subset of $L_0 \setminus I$, and hence of L_0, which contradicts (*).

If L_0 is a singleton, then $L_0 \subset \mathcal{B}^*$. Otherwise, L_0 is a nondegenerate continuum, and in particular a complete metric space. Property (**) implies that L_0 can be written as the union of $L_0 \cap \mathcal{B}^*$ with countably many nowhere dense subsets; if $L_0 \cap \mathcal{B}^*$ was countable, this would violate the Baire category theorem.

In the singleton case, write $L_0 = L \cap \mathcal{B}^* = \{c_0\}$. Let I be the set of indices i with $G_i \cap L \neq \emptyset$. Let us assume that $I \neq \emptyset$, as otherwise there is nothing to prove. By reordering, we may also assume that $I = \{1, \ldots, k\}$, where $0 \leq k \leq n$, and that furthermore G_1 is the last parameter ray which was completely removed in the construction of L_0, G_2 is the one completely removed before that, etc. By construction, the accumulation set L_1 of G_1 is contained in L_0, and hence G_1 lands at c_0. In fact, we inductively get

$$(1) \qquad L_i \subset \{c_0\} \cup \bigcup_{j=1}^{i-1} G_j.$$

It follows that every component of $\tilde{\mathbb{C}} \setminus L$ contains an interval of $\overline{\mathcal{S}} \setminus I$, and hence infinitely many parameter rays, as claimed.

(One way of seeing this is to recall that $\tilde{\mathbb{C}}$ is homeomorphic to the unit disk in $\mathbb{R}^2$. It follows from (1) and Janiszewski's theorem (see [P, Page 2] or [Ne, Theorem V.9·1·2]) that L, considered as a subset of $\mathbb{R}^2$ in this manner, does not separate the plane. So every component of $\tilde{\mathbb{C}} \setminus L$ must intersect the boundary of $\tilde{\mathbb{C}}$ in $\mathbb{R}^2$, i.e. $\overline{\mathcal{S}}$, as required.) △

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Proof of Theorem 7. (Properties of fibers.) We just proved the fact that fibers are either trivial or uncountable. Also, we proved that every parameter ray has some accumulation point in $\mathcal{B}^*$, and hence that every extended fiber intersects $\mathcal{B}^*$. The fact that extended fibers are connected was shown above in Lemma 18. ∎

Proof of Theorem 11. (Trivial fibers and landing of rays.) Let Y be a fiber, and suppose that every point of Y is the landing point of a parameter ray. By Lemma 17, this means that Y is finite. Hence, by Theorem 19, Y is trivial.

The converse follows directly from Theorem 19. ∎

Finally, let us prove the two remaining theorems, which deal exclusively with the Mandelbrot set $\mathcal{M}$.

Proof of Theorem 10. (Trivial fibers and local connectivity.) It is easy to see that triviality of a fiber Y in the Mandelbrot set implies local connectivity of $\mathcal{M}$ at Y. Indeed, as noted above, Y can be written as the nested intersection of countably many connected closed subsets of $\mathcal{B}$, each of which is a neighborhood of Y. (Compare [S4, Proposition 4.5].)

For the converse direction, suppose that $\mathcal{M}$ is locally connected at every point of Y. Let $z_0 \in Y$ be an accumulation point of some parameter ray G. Then there is a sequence C_k of cross-cuts of the domain $W := \hat{\mathbb{C}} \setminus \mathcal{M}$ (i.e. C_k is a Jordan arc intersecting $\mathcal{M}$ only in its two endpoints) with the following properties.

- C_k separates ∞ from all points on G with sufficiently small potential.
- The arcs C_k converge to $\{z_0\}$ in the Hausdorff distance.

Let W_k be the component of $W \setminus C_k$ not containing ∞. Then $I_G := \bigcap_k \overline{W_k}$ is the *prime end impression* of the parameter ray G. We will show that $I = \{z_0\}$.

Indeed, let $\varepsilon > 0$. Since $\mathcal{M}$ is locally connected at z_0, we can find a connected neighborhood K of z_0 in $\mathcal{M}$ of diameter less than ε, for any $\varepsilon > 0$. Since closures of connected sets are connected, we may assume that K is closed. For sufficiently large k, the arc C_k is a crosscut of K, and is contained in the disk of radius ε around z_0. It follows that diam $W_k \leq \varepsilon$, and hence diam $I \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $I = \{z_0\}$, as claimed.[6]

The boundary of any fiber Y is contained in the union of the prime end impressions corresponding to the parameter rays accumulating at Y. By Lemma 17, there are only

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[6]In the terminology of prime ends, we have just shown that the complement of a simply connected domain cannot be locally connected at any principal point of a nontrivial prime end impression. Even more is true: a prime end impression can contain at most two points in which the complement of the domain is locally connected; compare [R3].

finitely many such parameter rays. As we have just shown, for each of these rays the prime end impressions consist of a single point. The boundary of fiber Y is thus finite. Since Y is connected, it follows that Y is trivial as claimed. ∎

Proof of Theorem 4. (MLC implies density of hyperbolicity.) By Theorem 10, local connectivity of the Mandelbrot set is equivalent to triviality of fibers; by Theorem 9, triviality of fibers implies density of hyperbolicity. ∎

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